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Transport coefficients of the relativistic degenerate electron gas in a strong magnetic field

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Abstract. The transport coefficients for a relativistic degenerate electron plasma embedded in a strong magnetic field are calculated in a relaxation time approximation for the case of a collision dominated plasma. The heat conduction gives rise to two thermal conduction coefficients (parallel and transverse) plus a diffusion term. The two usual viscosity coefficients (bulk and shear) split into five viscosity coefficients (bulk, parallel, transverse, cross shear, plus bulk/shear). Transverse and longitudinal electric conductivities are also calculated. The model depends on a relaxation time which has to be evaluated according to the specific physical situation under consideration. The techniques used throughout this paper are those of the covariant Wigner distributions studied elsewhere.

1. Introduction

In many astrophysical systems relativistic quantum plasmas should be dealt with: white dwarfs, magnetosphere of pulsars, etc. Moreover strong (or moderately strong) magnetic fields also have to be considered in such systems: 10^6 to 10^8 G in the case of magnetic white dwarfs; 10^{12} G in the case of pulsars. Therefore a general study of relativistic quantum plasmas is an absolute necessity—specially when a magnetic field is present—in order to have a clear understanding of a large class of astrophysical phenomena.

In another paper (Dominguez Tenreiro and Hakim 1977a, to be referred to as I), some methods of relativistic quantum kinetic theory have been derived and discussed, and we first recall briefly a few results necessary for what follows in this paper.

The basic ingredient is the *quantum distribution function* defined by

$$F(x, p)=\frac{1}{(2\pi)^4}\int \mathrm{d}_4R\, \exp(-\mathrm{i}pR)\langle\bar{\psi}(x+\tfrac{1}{2}R)\otimes\psi(x-\tfrac{1}{2}R)\rangle \tag{1.1}$$

where ψ is a spin-$\frac{1}{2}$ quantum field describing the electron–positron field. In equation (1.1) the angular brackets represent a quantum statistical average:

$$\langle\bar{\psi}(x+\tfrac{1}{2}R)\otimes\psi(x-\tfrac{1}{2}R)\rangle\equiv\mathrm{Tr}(\rho\bar{\psi}(x+\tfrac{1}{2}R)\otimes\psi(x-\tfrac{1}{2}R)) \tag{1.2}$$

where ρ is the *density operator* describing the statistical state of the system. Note that F is a 4×4 matrix. In fact in many situations the effect of spin can be neglected (e.g. when $\mu B \ll kT$) and instead of the unusual quantum distribution (1.1) (or, equivalently, of the relativistic Wigner function (see I)) one can use a simpler, scalar distribution function, defined by (I)

$$f(x, p)=\frac{1}{(2\pi)^4}\int \mathrm{d}_4R\ \exp(-\mathrm{i}pR)\langle\bar{\psi}(x+\tfrac{1}{2}R)\,.\,\psi(x-\tfrac{1}{2}R)\rangle. \tag{1.3}$$

This last distribution will be used in all that follows. One can show (I) that it possesses the usual properties of a one-particle relativistic distribution function and, in particular, the four-current of the system and its energy-momentum tensor are given by the usual relations (I)

$$J^{\mu}(x)=\frac{1}{m}\int \mathrm{d}_4p\, p^{\mu}f(x, p) \tag{1.4}$$

$$T^{\mu\nu}(x)=\frac{1}{m}\int \mathrm{d}_4p\, p^{\mu}p^{\nu}f(x, p). \tag{1.5}$$

As a useful example, necessary in the following, we obtained the quantum distribution function for an electron fluid embedded in a strong magnetic field as (I)†

$$\begin{aligned} f_{\mathrm{eq}}(p)=\frac{1}{(2\pi)^3}\bigg(&\frac{\delta(p_0-E_0)}{1+\exp[\beta(E_0-\epsilon_{\mathrm{F}})]}\frac{m}{E_0}L_0(2w^2)\exp(-w^2)\\ &+\frac{1}{2}\sum_{n=0}^{\infty}\frac{m}{E_n}(-1)^n\frac{\delta(p_0-E_n)\exp(-w^2)}{1+\exp[\beta(E_{n+1}-\epsilon_{\mathrm{F}})]}(L_n(2w^2)-L_{n+1}(2w^2))\bigg). \end{aligned} \tag{1.6}$$

In equation (1.6) the E_n are directly related to the eigen-energies of an electron in an external magnetic field h:

$$E_n=\left(m^2+p_{\|}^2+\frac{m^2h}{H_{\mathrm{c}}}2n\right)^{1/2}; \qquad n=0, 1, 2, \ldots \tag{1.7}$$

($H_{\mathrm{c}}=4{\cdot}414\times10^{13}$ G), where $p_{\|}$ is the momentum parallel to the magnetic field. In equation (1.6) L_n is a Laguerre polynomial of order n, ϵ_{F} is the Fermi energy of the system and w^2 is given by

$$w^2=(p_1^2+p_2^2)/(|e|h/m) \tag{1.8}$$

($|e|$ being the charge of the electron) and is essentially connected with the transverse momentum of an electron in the magnetic field. In the derivation of equation (1.6) not only have we neglected interactions between the electrons but also the contributions of the positrons. As to this last approximation it is not difficult to add their contribution and furthermore in practical situations it is negligible (I). One more remark should be made: $f_{\mathrm{eq}}(p)$ is a Lorentz invariant written in the reference frame where the system is at rest and where the external electromagnetic field is purely magnetic.

These definitions would be useless if we could not give, derive or assume a (covariant) kinetic equation for $f(x, p)$. Such a kinetic equation first depends on the whole body of approximations or assumptions effected and next on the future use of $f(x, p)$.

In this paper we want to derive the transport coefficients of the relativistic degenerate electron gas embedded in a strong magnetic field for at least two reasons (briefly discussed in I): (i) it yields the most general form of the hydrodynamical equations of a relativistic charged fluid in the presence of a magnetic field; and (ii) it provides all those

† There is a misprint in I. The correct expression reads as in equation (1.6).

dissipative processes so important in relativistic astrophysics (they affect e.g. helium production in the early universe, the damping of magnetic fields, the stability of dense stars, etc). To this somewhat modest end we use the covariant generalisation of the BGK equation (Bhatnagar *et al* 1954) discussed in I:

$$p^{\lambda}\,\partial_{\lambda}f(x,p)-eF^{\mu}{}_{\lambda}p^{\lambda}\frac{\partial}{\partial p^{\mu}}f(x,p)=-u_{\mu}p^{\mu}\left(\frac{f(x,p)-f_{\mathrm{eq}}(p)}{\tau}\right) \tag{1.9}$$

(τ being the relaxation time (I); u^{μ} being the average four-velocity of the system; and $F^{\mu}{}_{\lambda}$ being the external magnetic field). This equation can be solved with the help of the Chapman–Enskog expansion which, at order one, is given by

$$f_{(1)}(x,p)\simeq-\frac{\tau}{u_{\lambda}p^{\lambda}}\left(p^{\lambda}\,\partial_{\lambda}f_{\mathrm{eq}}(x,p)+eF^{\mu}{}_{\lambda}p^{\lambda}\frac{\partial}{\partial p^{\mu}}f_{\mathrm{eq}}(x,p)\right) \tag{1.10}$$

where the dependence of f_{eq} on x occurs only through the *macroscopic* quantities u^{μ}, β, ϵ_{F} (I).

Although rough, the approximate calculations (relaxation time approximation, neglect of spin, etc) presented in this paper are sufficient when dealing with points (i) and (ii) above (I). In what follows we find that the determination of some transport coefficients is somewhat ambiguous as is the case in the non-relativistic and non-quantum limits when a magnetic field is present (Clemmow and Dougherty 1969). Nevertheless the relevant quantities, when dealing with dissipative effects, are unambiguous (up to the higher-order terms in the Chapman–Enskog expansion used in these papers): they are essentially the off-equilibrium parts of the four-current and of the energy–momentum tensor.

The calculated transport coefficients generally split into *transverse* and *longitudinal* parts: this is the case for heat or electrical conductivities or shear viscosities. However, in the shear part (i.e. the traceless part) of the off-equilibrium part of the energy–momentum tensor cross effects show up.

In this article use is made of the notation in I and the organisation of the paper is as follows. In § 2 the calculation of heat conduction coefficients is effected while in § 3 viscous effects are studied. In § 4 expressions for the electrical conductivities are given and discussed. Section 5 is devoted to comments on the results obtained.

1.1. *Notation*

Unless explicitly stated x, p, R, etc are four-vectors, the metric used having the signature $+---$. In what follows we use the notation

$$\Delta^{\mu\nu}=g^{\mu\nu}-u^{\mu}u^{\nu};\qquad \pi^{\mu\nu}=\Delta^{\mu\nu}+n^{\mu}n^{\nu},$$

where $g^{\mu\nu}$ is the usual Minkowski space–time metric tensor and where n^{μ} has the following properties (I):

$$n^{\mu}n_{\mu}=-1;\qquad n^{\mu}u_{\mu}=0;\qquad u^{\mu}u_{\mu}=1$$

$$F^{\mu\nu}=h\epsilon^{\mu\nu\alpha\beta}n_{\alpha}u_{\beta}.$$

2. Heat conduction

Let us first recall that in the description of Landau and Lifschitz (1975) of the relativistic fluid the four-current contains two parts, one representing the *streaming* of particles, the other being the *heat flux*, i.e.

$$J^\mu(x) = n_{\mathrm{eq}} u^\mu - q^\mu \tag{2.1}$$

with

$$n_{\mathrm{eq}} u^\mu \equiv J^\mu_{\mathrm{eq}} \tag{2.2}$$

$$u_\mu q^\mu = 0. \tag{2.3}$$

In this type of description there are no longer any heat conduction terms in the energy–momentum tensor and we show in § 3 that this is actually the case. Since, by definition

$$J^\mu(x) = J^\mu_{\mathrm{eq}}(x) + \tau J^\mu_{(1)}(x) + \mathrm{O}(\tau^2), \tag{2.4}$$

it follows that

$$q^\mu = -\tau J^\mu_{(1)} + \mathrm{O}(\tau^2). \tag{2.5}$$

Now multiplying equation (1.6) by p^α/m, integrating over p and using equation (4.19) of I, one gets finally†

$$\tau J^\alpha_{(1)} = -\frac{\tau}{m}(\partial_\lambda E^{\alpha\lambda} + \partial_\beta u_\lambda S^{\alpha\beta\lambda} + eF^{\mu\lambda} G^\alpha_{\mu\lambda}) \tag{2.6}$$

where we have set

$$E^{\alpha\lambda} = \int \mathrm{d}_4 p \frac{p^\alpha p^\lambda}{u_\sigma p^\sigma} f_{\mathrm{eq}}(x, p) \tag{2.7}$$

$$S^{\alpha\beta\lambda} = \int \mathrm{d}_4 p \frac{p^\alpha p^\beta p^\lambda}{(u_\sigma p^\sigma)^2} f_{\mathrm{eq}}(x, p) \tag{2.8}$$

$$G^\alpha_{\mu\lambda} = \int \mathrm{d}_4 p \frac{\partial}{\partial p^\mu}\left(\frac{p^\alpha p_\lambda}{u_\sigma p^\sigma}\right) f_{\mathrm{eq}}(x, p). \tag{2.9}$$

Because of the fact that $f_{\mathrm{eq}}(x, p)$ depends on the only four-vectors u^μ and n^μ, which are orthogonal to $F^{\mu\nu}$ and because of the antisymmetry of this tensor, it is easy to realise that the third term in equation (2.6) does not give any contribution to the off-equilibrium part of the current, say to $J^\mu_{(1)}(x)$. One can also check that the matching condition (4.15) of I is satisfied as is obvious in the forms given below for $J^\mu_{(1)}(x)$. Both the presence of τ and of gradients in equation (2.6) indicate its dissipative nature as it should be. However, only the first term can contain gradients of temperature and hence it will be used directly in the calculation of heat conduction coefficients.

Let us now calculate these coefficients. Since the tensors $E^{\alpha\lambda}$ and $S^{\alpha\beta\lambda}$ are completely symmetrical their general forms are‡

$$E^{\alpha\lambda} = E_1 u^\alpha u^\lambda - E_2 \pi^{\alpha\lambda} + E_3 n^\alpha n^\lambda + E_4(u^\alpha n^\lambda + u^\lambda n^\alpha) \tag{2.10}$$

$$S^{\alpha\beta\lambda} = s_1 u^\alpha u^\beta u^\lambda + 3s_2 u^{(\alpha}\pi^{\beta\lambda)} - 3s_3 u^{(\alpha} n^\beta n^{\lambda)} + 3s_4 n^{(\alpha}\pi^{\beta\lambda)} + 3s_5 n^{(\alpha} u^\beta u^{\lambda)} + s_6 n^\alpha n^\beta n^\lambda, \tag{2.11}$$

† From now on we drop the terms $\mathrm{O}(\tau^2)$ which are implicitly involved in our subsequent equations.

‡ The factor 3 is not a numerical factor but only indicates the number of terms present in the permutation (with repetitions) of the indices included between parentheses. In the following we also use the convention that symmetrization over indices (via parentheses) only applies to *free* indices. For instance, $A^{(\alpha\sigma\beta)}B_\sigma = A^{\alpha\sigma\beta}B_\sigma + A^{\beta\sigma\alpha}B_\sigma$.

since $f_{eq}(x, p)$ depends on the only macroscopic four-vectors u^α and n^α. In equations (2.10) and (2.11) the coefficients E_i $(i = 1, \ldots, 4)$ and s_i $(i = 1, \ldots, 6)$ depend only on the macroscopic quantities n_{eq}, β, h. The symmetry of $f_{eq}(x, p)$ under reflexions along the magnetic field (see equation (3.28) of I) leads to

$$E_4 = s_4 = s_5 = s_6 = 0. \tag{2.12}$$

Note also the relation

$$u_\lambda S^{\alpha\beta\lambda} = E^{\alpha\beta} \tag{2.13}$$

which yields

$$s_1 = E_1, \qquad s_2 = -E_2, \qquad s_3 = -E_3. \tag{2.14}$$

Moreover one has

$$E_1 = E^{\alpha\beta} u_\alpha u_\beta = \int \mathrm{d}_4 p\, u_\alpha p^\alpha f_{eq}(x, p) \equiv n_{eq} m \tag{2.15}$$

$$E_3 = -E^{\alpha\beta} n_\alpha n_\beta = -\int \mathrm{d}_4 p \frac{(n_\alpha p^\alpha)^2}{u_\sigma p^\sigma} f_{eq}(x, p) \tag{2.16}$$

$$E_2 = -\tfrac{1}{2} E^{\alpha\beta} \pi_{\alpha\beta} = -\tfrac{1}{2} \int \mathrm{d}_4 p \left(\frac{p_\alpha p^\alpha - (u_\alpha p^\alpha)^2 + (n_\alpha p^\alpha)^2}{u_\sigma p^\sigma} \right) f_{eq}(x, p). \tag{2.17}$$

Finally the heat flux four-vector q^α can be rewritten as†

$$\begin{aligned} \tau J^\alpha_{(1)} = -\frac{\tau}{m} \{ & u^\alpha [\dot{E}_2 + (E_2 - E_3) u_\lambda n'^\lambda] + n^\alpha [(E_3 - E_2)' + (E_3 - E_2) v + (E_3 - E_2) n^\lambda \dot{u}_\lambda] \\ & - \partial^\alpha E_2 + \dot{u}^\alpha E_1 + (E_3 - E_2) n'^\alpha \} \end{aligned} \tag{2.18}$$

(with $n'^\alpha \equiv n^\rho\, \partial_\rho n^\alpha$), where use has been made of the conservation of charge

$$\partial_\mu (n_{eq} u^\mu) = 0 \tag{2.19}$$

or

$$\dot{n}_{eq} + n_{eq} \theta = 0 \qquad (\text{with } \theta \equiv \partial_\mu u^\mu). \tag{2.20}$$

Also it is easy to check that the orthogonality relation (2.3) is satisfied. It follows that since q^α is, in the proper space, orthogonal to u^α it can be decomposed into two parts, one orthogonal to n^α and the other parallel to it, in the form

$$q^\alpha = q^\alpha_\parallel + q^\alpha_\perp \tag{2.21}$$

with

$$\begin{aligned} q^\alpha_\parallel &= -n^\alpha n_\beta q^\beta \\ q^\alpha_\perp &= \pi^\alpha_\beta q^\beta. \end{aligned} \tag{2.22}$$

The *longitudinal heat flux* $q^\alpha_\parallel$ is given by

$$q^\alpha_\parallel = -\frac{\tau}{m} n^\alpha [E'_3 + (E_3 - E_2) v + (E_3 - E_2) n^\lambda \dot{u}_\lambda - n_{eq} m n^\lambda \dot{u}_\lambda] \equiv -\tau J^\alpha_{(1)\parallel} \tag{2.23}$$

† Remember that the prime designates the derivative parallel to n^μ $(A' \equiv n^\alpha\, \partial_\alpha A)$ while the dot designates the derivative parallel to u^μ $(\dot{A} \equiv u^\alpha\, \partial_\alpha A)$; moreover $v \equiv \partial_\alpha n^\alpha$.

while the *transverse heat flux* $q_\perp^\alpha$ is

$$q_\perp^\alpha = -\frac{\tau}{m}\pi^{\alpha\beta}[-\partial_\beta E_2 + \dot{u}_\beta n_{\text{eq}} m + (E_3 - E_2)n'_\beta] \equiv -\tau J^\alpha_{(1)\perp}. \tag{2.24}$$

To these heat fluxes correspond *two* heat conduction coefficients, respectively $\lambda_\parallel$ and $\lambda_\perp$, defined through

$$Q_\parallel^\alpha = -\lambda_\parallel n^\alpha n^\nu(\beta^{-1}\dot{u}_\nu - \partial_\nu \beta^{-1}) \tag{2.25}$$

$$Q_\perp^\alpha = \lambda_\perp \pi^{\alpha\nu}(\beta^{-1}\dot{u}_\nu - \partial_\nu \beta^{-1}). \tag{2.26}$$

These Q^α are in fact the *true* heat fluxes which include the so called Eckart inertial term $\lambda\beta^{-1}\dot{u}_\nu$ (Eckart 1940) (or rather their respective projections) and, as we see below, $q^\alpha \neq Q^\alpha$. The definitions (2.25) and (2.26) do not change the values of the heat condition coefficients: they only change the definitions of heat fluxes by including into them the inertia and momentum of heat. We have

$$\beta^2\lambda_\parallel = \text{numerical coefficient of } (-n^\alpha n^\nu\, \partial_\nu \beta) \tag{2.27}$$

$$\beta^2\lambda_\perp = \text{numerical coefficient of } (\pi^{\alpha\nu}\, \partial_\nu \beta). \tag{2.28}$$

In order to obtain these coefficients we first note that

$$\partial_\mu E_i = \partial_\mu \beta \frac{\partial E_i}{\partial \beta} + \partial_\mu n_{\text{eq}} \frac{\partial E_i}{\partial n_{\text{eq}}} + \partial_\mu |h|^2 \frac{\partial E_i}{\partial |h|^2} \tag{2.29}$$

($i = 1, 2, 3$). We now express $\partial_\mu n_{\text{eq}}$ as a function of $\partial_\mu \beta$ with the help of the energy–momentum conservation relation at order zero in τ: it turns out that the relation between these two quantities has the form

$$\Gamma_1^{\alpha\nu}\,\partial_\alpha \beta + \Gamma_2^{\alpha\nu}\,\partial_\alpha n_{\text{eq}} + \Gamma_3^{\alpha\nu}\,\partial_\alpha |h|^2 + K^\nu = 0$$

with

$$\Gamma_1^{\alpha\nu} = u^\alpha u^\nu \frac{\partial \rho}{\partial \beta} - \pi^{\alpha\nu}\frac{\partial P_\perp}{\partial \beta} + n^\alpha n^\nu \frac{\partial P_\parallel}{\partial \beta} \tag{2.30}$$

$$\Gamma_2^{\alpha\nu} = u^\alpha u^\nu \frac{\partial \rho}{\partial n_{\text{eq}}} - \pi^{\alpha\nu}\frac{\partial P_\perp}{\partial n_{\text{eq}}} + n^\alpha n^\nu \frac{\partial P_\parallel}{\partial n_{\text{eq}}} \tag{2.31}$$

$$\Gamma_3^{\alpha\nu} = u^\alpha u^\nu \frac{\partial \rho}{\partial |h|^2} - \pi^{\alpha\nu}\frac{\partial P_\perp}{\partial |h|^2} + n^\alpha n^\nu \frac{\partial P_\parallel}{\partial |h|^2} \tag{2.32}$$

$$K^\alpha = (\rho + P_\perp)(\dot{u}^\alpha + \theta u^\alpha) + (P_\parallel - P_\perp)(v n^\alpha + n'^\alpha). \tag{2.33}$$

It follows that

$$\partial_\alpha n_{\text{eq}} = -(\Gamma_2^{-1})_{\alpha\sigma}\Gamma_1^{\sigma\rho}\,\partial_\rho \beta - (\Gamma_2^{-1})_{\alpha\sigma}\Gamma_3^{\sigma\rho}\,\partial_\rho |h|^2 - (\Gamma_2^{-1})_{\alpha\sigma}K^\sigma \tag{2.34}$$

and one easily finds†

$$(\Gamma_2^{-1})^{\alpha\beta} = \left(\frac{\partial \rho}{\partial n_{\text{eq}}}\right)^{-1} u^\alpha u^\beta - \left(\frac{\partial P_\perp}{\partial n_{\text{eq}}}\right)^{-1}\pi^{\alpha\beta} + \left(\frac{\partial P_\parallel}{\partial n_{\text{eq}}}\right)^{-1} n^\alpha n^\beta. \tag{2.35}$$

† $(\Gamma_2^{-1})^{\alpha\beta}$ necessarily has the form $au^\alpha u^\beta + b\pi^{\alpha\beta} + cn^\alpha n^\beta$. The three unknown coefficients (a, b, c) are obtained from the conditions $\Gamma_2^{\alpha\sigma}(\Gamma_2^{-1})_{\sigma\beta} = g^\alpha_\beta$.

Substituting equation (2.35) into equation (2.29) and the result into equations (2.25) and (2.26), we find

$$\tau J^{\sigma}_{(1)\parallel}=\frac{\tau}{m}n^{\sigma}\left[\left(-\frac{\partial E_3}{\partial n_{\text{eq}}}n^{\alpha}(\Gamma_2^{-1})_{\alpha\nu}\Gamma_1^{\nu\rho}+\frac{\partial E_3}{\partial\beta}n^{\rho}\right)\partial_\rho\beta+\frac{\partial E_3}{\partial|h|^2}|h|^{2'}\right.$$
$$\left.+(E_3-E_2)v+(E_3-E_2+mn_{\text{eq}})n^{\lambda}\dot{u}_{\lambda}-(\Gamma_2^{-1})_{\alpha\rho}n^{\alpha}K^{\rho}\frac{\partial E_3}{\partial n_{\text{eq}}}\right] \tag{2.36}$$

$$\tau J^{\sigma}_{(1)\perp}=-\frac{\tau}{m}\pi^{\sigma\alpha}\left[\left((\Gamma_2^{-1})_{\alpha\nu}\Gamma_1^{\nu\rho}\frac{\partial E_2}{\partial n_{\text{eq}}}-g^{\rho}_{\alpha}\right)\partial_\rho\beta-\frac{\partial E_2}{\partial|h|^2}\partial_\alpha|h|^2\right.$$
$$\left.+mn_{\text{eq}}\dot{u}_\alpha+(E_3-E_2)n'_\alpha-(\Gamma_2^{-1})_{\alpha\rho}K^{\rho}\frac{\partial E_2}{\partial n_{\text{eq}}}\right]. \tag{2.37}$$

Using equations (2.31) and (2.35), the definition (2.33) of K^{ρ}, equations (2.25) and (2.26), equations (2.27) and (2.28), the heat conduction coefficients are found to be

$$\lambda_{\parallel}=\frac{\tau}{m}\beta^{-2}\left\{\left[\frac{\partial P_{\parallel}}{\partial\beta}\left(\frac{\partial P_{\parallel}}{\partial n_{\text{eq}}}\right)^{-1}\frac{\partial E_3}{\partial n_{\text{eq}}}\right]-\frac{\partial E_3}{\partial\beta}\right\} \tag{2.38}$$

$$\lambda_{\perp}=\frac{\tau}{m}\beta^{-2}\left\{\left[\frac{\partial P_{\perp}}{\partial\beta}\left(\frac{\partial P_{\perp}}{\partial n_{\text{eq}}}\right)^{-1}\frac{\partial E_2}{\partial n_{\text{eq}}}\right]-\frac{\partial E_2}{\partial\beta}\right\}. \tag{2.39}$$

We now have to come back to the physical interpretation of some definitions given above. First we want to stress that the only physically meaningful quantity is the off-equilibrium part $J^{\mu}_{(1)}$ (or equivalently $J^{\mu}_{(1)\parallel}$ and $J^{\mu}_{(1)\perp}$) of the four-current J^{μ}. The fact that it is called the heat flux or that only Q^{μ} is called the 'true' heat flux is somewhat arbitrary and refers *only* to the presence of *spatial gradients of the temperature* in its formal expression. However, *if* we still insist that heat fluxes are defined through equations (2.25) and (2.26)—and these definitions are quite natural since they put forward the existence of momentum and inertia of heat—then it remains to interpret the other terms, i.e. $J^{\mu}_{(1)}-Q^{\mu}$. In fact they should *probably* be looked at as *diffusion terms.* To explain this point more precisely and in order to obtain a deeper insight into their physical interpretation, instead of eliminating the gradients of n_{eq} with the help of the zeroth-order energy-momentum conservation relation, we could also eliminate the gradients of $|h|^2$. Doing so, we should find

$$\bar{\lambda}_{\perp}\propto\frac{\partial E_2}{\partial\beta}-\left(\frac{\partial P_{\perp}}{\partial|h|^2}\right)^{-1}\frac{\partial E_2}{\partial|h|^2}\frac{\partial P_{\perp}}{\partial\beta} \tag{2.40}$$

$$\bar{\lambda}_{\parallel}\propto\frac{\partial E_3}{\partial\beta}-\left(\frac{\partial P_{\parallel}}{\partial|h|^2}\right)^{-1}\frac{\partial E_3}{\partial|h|^2}\frac{\partial P_{\parallel}}{\partial\beta} \tag{2.41}$$

$$D_{\perp}\propto\frac{\partial E_2}{\partial n_{\text{eq}}}-\left(\frac{\partial P_{\perp}}{\partial|h|^2}\right)^{-1}\frac{\partial E_2}{\partial|h|^2}\frac{\partial P_{\perp}}{\partial n_{\text{eq}}} \tag{2.42}$$

$$D_{\parallel}\propto\frac{\partial E_3}{\partial n_{\text{eq}}}-\left(\frac{\partial P_{\parallel}}{\partial|h|^2}\right)^{-1}\frac{\partial E_3}{\partial|h|^2}\frac{\partial P_{\parallel}}{\partial n_{\text{eq}}}. \tag{2.43}$$

The first two equations represent the new thermal conductivity coefficients, while $D_{\parallel}$ and $D_{\perp}$ are respectively the parallel and transverse diffusion coefficients. The ambiguity in the definition of heat conduction coefficients is by no means new and also occurs in classical situations when a magnetic field is present (see e.g. Clemmow and Dougherty 1969). In the classical case one usually finds a third heat conduction coefficient. Here this is not the case because of the aforementioned symmetry of $f_{\mathrm{eq}}(p)$ under reflexions along the magnetic field (see § 5).

To summarise this brief discussion $J^{\mu}_{(1)}$ must be considered as the heat flux four-vector although its detailed decomposition leaves place to some arbitrariness.

Let us also add that the heat conduction coefficients (2.38) and (2.39) tend to the same value λ

$$\lim_{h\to 0} \lambda_{\parallel} = \lim_{h\to 0} \lambda_{\perp} = \lambda,$$

when the magnetic field is switched off. This common value is the one already given by Anderson and Witting (1974) in the absence of magnetic field.

3. Viscous stress

We first notice that the viscous stress tensor has exactly the same form either in Eckart's relativistic hydrodynamics or in that of Landau and Lifschitz so that there is no problem of passage from one form to the other. Moreover we have already remarked in I that the matching conditions: (i) are imposed (equations (4.8)–(4.10) of I) by the covariant BGK equations; and (ii) yield the same results as those of Landau and Lifschitz (equation (4.12) of I) (see below).

Let us now calculate the first-order correction $T^{\alpha\beta}_{(1)}$,

$$T^{\alpha\beta} = T^{\alpha\beta}_{\mathrm{eq}} + \tau T^{\alpha\beta}_{(1)} + \mathrm{O}(\tau^2) \tag{3.1}$$

to the energy–momentum tensor. To this end we multiply equation (1.10) by $p^{\alpha}p^{\beta}/m$ and integrate over the p variables. We then get

$$T^{\alpha\beta}_{(1)} = -(\partial_{\sigma}S^{\alpha\beta\sigma} + \partial_{\sigma}u_{\lambda}Q^{\alpha\beta\sigma\lambda}) \tag{3.2}$$

where

$$S^{\alpha\beta\sigma} = \frac{1}{m}\int \mathrm{d}_4p \frac{p^{\alpha}p^{\beta}p^{\sigma}}{u_{\lambda}p^{\lambda}} f_{\mathrm{eq}}(x, p) \tag{3.3}$$

$$Q^{\alpha\beta\sigma\lambda} = \frac{1}{m}\int \mathrm{d}_4p \frac{p^{\alpha}p^{\beta}p^{\sigma}p^{\lambda}}{(u_{\lambda}p^{\lambda})^2} f_{\mathrm{eq}}(x, p); \tag{3.4}$$

and as in § 2 the term involving $F^{\mu\nu}$ does not contribute to $T^{\alpha\beta}_{(1)}$ and for the same reasons. The symmetric tensors $S^{\alpha\beta\sigma}$ and $Q^{\alpha\beta\sigma\lambda}$ can be decomposed as†

$$S^{\alpha\beta\sigma} = S_1u^{\alpha}u^{\beta}u^{\sigma} + 3S_2u^{(\alpha}\pi^{\beta\sigma)} - 3S_3u^{(\alpha}n^{\beta}n^{\sigma)} \tag{3.5}$$

and

$$\begin{aligned} Q^{\alpha\beta\sigma\lambda} = {} & Q_1u^{\alpha}u^{\beta}u^{\sigma}u^{\lambda} + 6Q_2\pi^{(\alpha\beta}u^{\sigma}u^{\lambda)} + 3Q_3\pi^{(\alpha\beta}\pi^{\sigma\lambda)} \\ & + Q_4n^{\alpha}n^{\beta}n^{\sigma}n^{\lambda} - 6Q_5\pi^{(\alpha\beta}n^{\sigma}n^{\lambda)} - 6Q_6u^{(\alpha}u^{\beta}n^{\sigma}n^{\lambda)}. \end{aligned} \tag{3.6}$$

In equations (3.5) and (3.6) the coefficients of terms involving an *odd* number of n^{α} vanish because of the invariance of $f_{\mathrm{eq}}(p)$ under reflexions along the magnetic field. The

† See second footnote to p 1528.

remaining coefficients S_i, as well as the Q_i, are easily expressed as integrals. Also we have

$$Q^{\alpha\beta\sigma\lambda}u_\sigma u_\lambda = T^{\alpha\beta}_{\rm eq} = S^{\alpha\beta\sigma}u_\sigma \tag{3.7}$$

which provides

$$S_1 = Q_1 = \rho, \qquad S_3 = Q_6 = -P_\parallel, \qquad S_2 = Q_2 = -P_\perp. \tag{3.8}$$

After some calculations using equations (3.2) to (3.8) we finally obtain

$$\begin{aligned}
-T^{\alpha\beta}_{(1)} = {} & u^\alpha u^\beta[\dot S_1 + (S_1 - S_2)\theta + (S_2 - S_3)n^\lambda u'_\lambda] \\
& - n^\alpha n^\beta[\dot S_3 + (S_3 + Q_5)\theta + (5Q_5 - Q_4 - 2Q_3)n^\lambda u'_\lambda] \\
& + \pi^{\alpha\beta}[\dot S_2 + (S_2 + Q_3)\theta + (Q_3 - Q_5)n^\lambda u'_\lambda] + u^{(\alpha}n^{\beta)}[-S'_3 + (S_2 - S_3)v \\
& + (S_2 - S_3 + Q_5 - Q_3)n^\lambda \dot u_\lambda] + u^{(\alpha}\dot u^{\beta)}(S_1 - S_2) \\
& + \pi^{(\alpha\sigma}\partial_\sigma u^{\beta)}(Q_3 + S_2) + \pi^{(\alpha\sigma}u^{\beta)}\partial_\sigma S_2 - n^{(\alpha}u'^{\beta)}(S_3 + Q_5) \\
& + n^{(\alpha}\partial^{\beta)}u_\lambda \, . \, n^\lambda(Q_3 - Q_5) + u^{(\alpha}n'^{\beta)}(S_2 - S_3) + n^{(\alpha}\dot n^{\beta)}(S_2 - S_3).
\end{aligned} \tag{3.9}$$

The matching conditions read

$$\dot S_1 + (S_1 - S_2)\theta + (S_2 - S_3)u_\lambda n'^\lambda = 0 \tag{3.10}$$

$$S'_3 + (S_3 - S_2)v + (S_1 - S_2)n_\lambda \dot u^\lambda = 0. \tag{3.11}$$

Using equations (3.8), (3.10) and (3.11), equation (3.9) can be rewritten as

$$\begin{aligned}
-T^{\alpha\beta}_{(1)} = {} & \pi^{\alpha\beta}[\dot S_2 + (S_2 + Q_3)\theta + (Q_3 - Q_5)n^\lambda u'_\lambda] \\
& - n^\alpha n^\beta[\dot S_3 + (S_3 + Q_5)\theta + (Q_4 + 3Q_5)n^\lambda u'_\lambda] \\
& + \pi^{(\alpha\sigma}[n^{\beta)}n^\lambda\ \partial_\sigma u_\lambda(Q_3 - Q_5) + \partial_\sigma u^{\beta)}(S_2 + Q_3)] - n^{(\alpha}n^\sigma\ \partial_\sigma u^{\beta)}(S_3 + Q_5) \\
& + u^{(\alpha}n^{\beta)}n^\lambda \dot u_\lambda(S_2 - S_3) + n^{(\alpha}\dot n^{\beta)}(S_2 - S_3).
\end{aligned} \tag{3.12}$$

From this last expression for $T^{\alpha\beta}_{(1)}$ it is easy to see that

$$u_\beta T^{\alpha\beta}_{(1)} = 0 \tag{3.13}$$

i.e., the Landau and Lifschitz matching conditions are also satisfied. Let us now split $T^{\alpha\beta}_{(1)}$ into a traceless part $\psi^{\alpha\beta}$ and a part proportional to $\Delta^{\alpha\beta}(u^\rho)$ since $T^{\alpha\beta}_{(1)}$ is in the proper space of u^ρ as shown by the form of the above equation (3.13):

$$\begin{aligned}
& T^{\alpha\beta}_{(1)} = \psi^{\alpha\beta} + \tfrac{1}{3}\Delta^{\alpha\beta}(u^\rho) \times T^{\alpha}_{(1)\alpha} \\
& \psi^\alpha{}_\alpha = 0.
\end{aligned} \tag{3.14}$$

We then get

$$\begin{aligned}
-T^{\alpha\beta}_{(1)} = {} & \pi^{\alpha\beta}[\dot S_2 + (S_2 + Q_3)\theta + (Q_3 - Q_5)n^\lambda u'_\lambda + \tfrac{1}{3}T^{\lambda}_{(1)\lambda}] \\
& - n^\alpha n^\beta[\dot S_3 + (S_3 + Q_5)\theta + (Q_4 + 3Q_5)n^\lambda u'_\lambda + \tfrac{1}{3}T^{\lambda}_{(1)\lambda}] \\
& + \pi^{(\alpha\sigma}[n^{\beta)}n^\lambda\ \partial_\sigma u_\lambda(Q_3 - Q_5) + \partial_\sigma u^{\beta)}(S_2 + Q_3)] - n^{(\alpha}n^\sigma\ \partial_\sigma u^{\beta)}(S_3 + Q_5) \\
& + u^{(\alpha}n^{\beta)}n^\lambda \dot u_\lambda(S_2 - S_3) + n^{(\alpha}\dot n^{\beta)}(S_2 - S_3) - \tfrac{1}{3}\Delta^{\alpha\beta}(u^\rho)T^{\lambda}_{(1)\lambda}
\end{aligned} \tag{3.15}$$

with

$$T^{\lambda}_{(1)\lambda} = -\{2\dot S_2 + \dot S_3 + \theta[4(S_2 + Q_3) + (S_3 + Q_5)] + n^\lambda u'_\lambda(2S_2 + 4Q_3 - 2S_3 - Q_5 + Q_4)\}. \tag{3.16}$$

As usual the traceless part of $T^{\alpha\beta}_{(1)}$ will provide the *shear viscosity* coefficients while the trace part will furnish the bulk viscosity coefficient. However, in order to obtain them it is preferable to use the projections of $\psi^{\alpha\beta}$ on the *two*† orthogonal projectors $\pi^{\alpha\beta}$, $-n^\alpha n^\beta$ instead of using its expression given by equation (3.15):

$$\psi^{\alpha\beta}=\pi^{\alpha\lambda}\pi^{\beta\rho}\psi_{\lambda\rho}+n^\alpha n^\beta(n^\lambda n^\rho\psi_{\lambda\rho})-\pi^{(\alpha\lambda}n^{\beta)}n^\rho\psi_{\lambda\rho}+u^\alpha u^\beta(u^\lambda u^\rho\psi_{\lambda\rho})$$
$$+\pi^{(\alpha\lambda}u^{\beta)}u^\rho\psi_{\lambda\rho}-n^{(\alpha}n^\lambda u^{\beta)}u^\rho\psi_{\lambda\rho} \tag{3.17}$$

where, of course†, the last three terms of this equation vanish identically. We find

$$-T^{\alpha\beta}_{(1)}=\pi^{\alpha\beta}[\dot S_2+(S_2+Q_3)\theta+(Q_3-Q_5)n^\lambda u'_\lambda+\tfrac{1}{3}T^\lambda_{(1)\lambda}]$$
$$-n^\alpha n^\beta[\dot S_3+(S_3+Q_5)\theta+(Q_4+3Q_5)n^\lambda u'_\lambda+\tfrac{1}{3}T^\lambda_{(1)\lambda}]$$
$$+(S_2+Q_3)\pi^{\alpha\mu}\pi^{\beta\rho}(\partial_\rho u_\mu+\partial_\mu u_\rho)+(S_3+Q_5)n^\alpha n^\beta n^\mu n^\rho(\partial_\rho u_\mu+\partial_\mu u_\rho)$$
$$-(S_3+Q_5)\pi^{(\alpha\mu}n^{\beta)}n^\rho(\partial_\rho u_\mu+\partial_\mu u_\rho)+(S_3-S_2)\pi^{(\alpha\mu}n^{\beta)}n^\rho(\partial_\rho u_\mu-\partial_\mu u_\rho)$$
$$-\tfrac{1}{3}\Delta^{\alpha\beta}(u^\rho)T^\lambda_{(1)\lambda}. \tag{3.18}$$

In this equation the last term is the bulk viscosity stress while the term before represents the effect of *vorticity* of the stream lines. This term also exists in the non-relativistic case (e.g. Kaufman 1960). The *five* remaining terms represent *shear* viscosity. However, before giving the expressions for the viscosity coefficients we have to evaluate the S_i occurring in equation (3.18). We have

$$\dot S_i=-n_{\text{eq}}\theta\frac{\partial S_i}{\partial n_{\text{eq}}}+\dot\beta\frac{\partial S_i}{\partial\beta}+|h^2|^{\cdot}\frac{\partial S_i}{\partial|h|^2} \tag{3.19}$$

where use has been made of equation (2.20). $\dot\beta$ is now calculated from the expression (3.36) given in I for $\dot S_1$ and reads

$$\dot\beta=\left(\frac{\partial S_1}{\partial\beta}\right)^{-1}\theta\left(n_{\text{eq}}\frac{\partial S_1}{\partial n_{\text{eq}}}-(S_1-S_2)\right)-\left(\frac{\partial S_1}{\partial\beta}\right)^{-1}\left((S_2-S_3)u_\lambda n'^\lambda+|h^2|^{\cdot}\frac{\partial S_1}{\partial|h|^2}\right). \tag{3.20}$$

As to $|h^2|^{\cdot}$ it can be calculated by using Maxwell's equation $\partial_\mu F^{\mu\nu *}=0$, a consequence of which is (see e.g. Lichnerowicz 1971)

$$\tfrac{1}{2}|h^2|^{\cdot}+\theta|h|^2+n^\lambda u'_\lambda|h|^2=0. \tag{3.21}$$

Finally it turns out that $\dot S_i$ can be written as

$$\dot S_i=A_i\theta+B_in^\lambda u'_\lambda \qquad (i=2,3) \tag{3.22}$$

with

$$A_i=\left[-n_{\text{eq}}\frac{\partial S_i}{\partial n_{\text{eq}}}+\left(\frac{\partial S_1}{\partial\beta}\right)^{-1}\frac{\partial S_i}{\partial\beta}\left(n_{\text{eq}}\frac{\partial S_1}{\partial n_{\text{eq}}}-(S_1-S_2)\right)\right]$$
$$+2|h|^2\left[-\frac{\partial S_i}{\partial|h|^2}+\frac{\partial S_i}{\partial\beta}\left(\frac{\partial S_1}{\partial\beta}\right)^{-1}\left(\frac{\partial S_1}{\partial|h|^2}\right)\right] \tag{3.23}$$

$$B_i=2|h|^2\left[-\frac{\partial S_i}{\partial|h|^2}+\frac{\partial S_i}{\partial\beta}\left(\frac{\partial S_1}{\partial\beta}\right)^{-1}\left(\frac{\partial S_1}{\partial|h|^2}-\frac{S_2-S_3}{2|h|^2}\right)\right] \tag{3.24}$$

† There is no need for the projector $u^\alpha u^\beta$ because of equation (3.13).

$(i=2,3)$. Now introducing the *shear tensor* $\sigma_{\mu\nu}$,

$$\sigma_{\mu\nu}=(\partial_\mu u_\nu+\partial_\nu u_\mu)-\tfrac{2}{3}\theta\,\Delta_{\mu\nu}(u^\rho), \tag{3.25}$$

the traceless part $\psi^{\alpha\beta}$ of the viscous stress tensor $T^{\alpha\beta}_{(1)}$ can be rewritten (without the vorticity term) as

$$\begin{aligned}-\psi^{\alpha\beta}=&(S_2+Q_3)(\pi^{\alpha\mu}\pi^{\beta\rho}-\tfrac{1}{2}\pi^{\alpha\beta}\pi^{\mu\rho})\sigma_{\mu\rho}-(S_3+Q_5)(\pi^{(\alpha\mu}n^{\beta)}n^\rho)\sigma_{\mu\rho}\\&+N_2[\tfrac{3}{2}(\tfrac{2}{3}n^\alpha n^\beta+\tfrac{1}{3}\pi^{\alpha\beta})(\tfrac{2}{3}n^\mu n^\rho+\tfrac{1}{3}\pi^{\mu\rho})]\sigma_{\mu\rho}+(M_2-\tfrac{1}{3}N_2)(\pi^{\alpha\beta}+2n^\alpha n^\beta)\theta\end{aligned} \tag{3.26}$$

where

$$M_2=-\tfrac{1}{3}[4(S_2+Q_3)+(S_3+Q_5)+2A_2+A_3]+2(S_2+Q_3)+A_2 \tag{3.27}$$

$$N_2=-\tfrac{1}{3}(2S_2+4Q_3-2S_3-Q_5+Q_4+2B_2+B_3)+(S_2+Q_3)+B_2+(Q_3-Q_5). \tag{3.28}$$

(Note also that for $h\to 0$, $N_2\to S_2+Q_3$.) Let us now identify the viscosity coefficients. In fact there is no unique way to define these coefficients as is obvious from the literature on the subject (see, e.g., Kaufman 1960, Braginskii 1965, Coope and Snider 1970, De Groot and Mazur 1962, Clemmow and Dougherty 1969) and the grouping of the components of $\partial_{(\mu}u_{\nu)}$ (or $\sigma_{\mu\nu}$) which serves to define them is largely a matter of taste or adaptation to experimental situations *or* symmetries of the problem under consideration. Nevertheless it should be noticed that the physically meaningful quantity is the viscous stress tensor and not necessarily its decomposition. In equation (3.26) we have used a covariant generalisation of Braginskii's (1965) decomposition:

$$\psi^{\alpha\beta}=-\eta_0W_0^{\alpha\beta}-\eta_1W_1^{\alpha\beta}-\eta_2W_2^{\alpha\beta}-\eta_3W_3^{\alpha\beta}-\eta_4W_4^{\alpha\beta}-\bar\eta\bar W^{\alpha\beta} \tag{3.29}$$

where the η_i and $\bar\eta$ are the viscosity coefficients and the $W_i^{\alpha\beta}$ are given by

$$W_0^{\alpha\beta}=\tfrac{3}{2}[(n^\alpha n^\beta+\tfrac{1}{3}\Delta^{\alpha\beta}(u^\rho))(n^\mu n^\rho+\tfrac{1}{3}\Delta^{\mu\rho}(u^\lambda))]\sigma_{\mu\rho} \tag{3.30}$$

$$W_1^{\alpha\beta}=(\pi^{\alpha\mu}\pi^{\beta\rho}-\tfrac{1}{2}\pi^{\alpha\beta}n^\mu n^\rho)\sigma_{\mu\rho} \tag{3.31}$$

$$W_2^{\alpha\beta}=-(\pi^{\alpha\mu}n^\beta n^\rho+\pi^{\beta\rho}n^\alpha n^\mu)\sigma_{\mu\rho} \tag{3.32}$$

$$W_3^{\alpha\beta}=\tfrac{1}{2}(\pi^{\alpha\mu}\epsilon^{\beta\gamma\rho}+\pi^{\beta\rho}\epsilon^{\alpha\gamma\mu})n_\gamma\sigma_{\mu\rho} \tag{3.33}$$

$$W_4^{\alpha\beta}=-(n^\alpha n^\mu\epsilon^{\beta\gamma\rho}+n^\beta n^\rho\epsilon^{\alpha\gamma\mu})n_\gamma\sigma_{\mu\rho} \tag{3.34}$$

$$\bar W^{\alpha\beta}=(\pi^{\alpha\beta}+2n^\alpha n^\beta)\theta \tag{3.35}$$

where

$$\epsilon^{\alpha\gamma\mu}\underset{\text{def}}{\equiv}\epsilon^{\alpha\gamma\mu\lambda}u_\lambda. \tag{3.36}$$

These $W_i^{\alpha\beta}$ are orthogonal and traceless. In our case we have

$$\eta_3=\eta_4=0 \tag{3.37}$$

because of the repeatedly mentioned symmetry of f_{eq} along the magnetic field direction which forbids the appearance of terms odd in n^α. Moreover it is easy to identify (3.30) with the third term of (3.26), (3.31) with the first term of (3.26) and (3.33) with its second term. This identification yields

$$\eta_0 = N_2 \tag{3.38}$$

$$\eta_1 = S_2 + Q_3 \tag{3.39}$$

$$\eta_2 = S_3 + Q_5. \tag{3.40}$$

The last term in equation (3.26) is to be identified with a viscous cross effect between shear and dilatation. It also exists in the classical case (see, e.g., De Groot and Mazur 1962) but not in Braginskii's decomposition due to the particular problem treated. This coefficient, conventionally called ζ, is given by

$$\zeta \equiv \bar{\eta} = M_2 - \tfrac{1}{3}N_2. \tag{3.41}$$

Before discussing the bulk term in equation (3.18) let us add a few words about the vorticity term occurring in the same equation, say

$$(S_3 - S_2)\pi^{(\alpha\mu}n^{\beta)}n^{\rho}(\partial_\rho u_\mu - \partial_\mu u_\rho). \tag{3.42}$$

(Recall that $S_3 - S_2 \equiv P_\perp - P_\parallel$.) This term also exists in the classical case (see, e.g., Kaufman 1960). Obviously it occurs because of the rotation of the charged particles around the magnetic field axis and it is to be expected that it has the maximum value at zero temperature and that it vanishes at high temperature: this is clear from the fact that it is proportional to $P_\perp - P_\parallel$, this quantity tending to zero as $T \to \infty$.

Let us now discuss the bulk effect contained in the last term of equation (3.18). Using equations (3.16), (3.19) and (3.21) it is found that $T^{\lambda}_{(1)\lambda}$ can be written as

$$-T^{\lambda}_{(1)\lambda} = \theta\Big(2A_2 + A_3 + 4(S_2 + Q_3) + (S_3 + Q_5) - (2S_2 + 4Q_3 - 2S_3 - Q_5 + Q_4) - \frac{1}{|h^2|}(2B_2 + B_3)\Big) - \frac{|h^2|}{2|h|^2}(2S_2 + 4Q_3 - 2S_3 - Q_5 + Q_4 + 2B_2 + B_3). \tag{3.43}$$

The first term of this equation provides the *bulk viscosity* coefficient η_v:

$$\eta_v = 2A_2 + A_3 + 2S_2 + 3S_3 - Q_4 + 2Q_5 - \frac{1}{|h|^2}(2B_2 + B_3) \tag{3.44}$$

whereas the second is a contribution to the pressure. Once more it should be noticed that this decomposition—and thus this definition—is largely arbitrary, the physical quantity being simply the dilatation term $-\frac{1}{3}T^{\lambda}_{(1)\lambda}\Delta^{\alpha\beta}(u^\rho)$. Also this reflects the arbitrariness of the decomposition of the energy–momentum tensor into a particle part and a magnetic field part: this pressure term could as well be included in the energy–momentum tensor of the magnetic field.

4. The conductivity tensor

Let us now look for the conductivity tensor $\Lambda^{\mu\alpha\beta}$ defined through

$$J^{\mu}_{\text{resp}} = \Lambda^{\mu\alpha\beta}F^{\text{ext}}_{\alpha\beta} \tag{4.1}$$

where $F^{\text{ext}}_{\alpha\beta}$ is a *weak*† external electric field (i.e. one has

$$\begin{aligned}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}_{\text{ext}}F^{\alpha\beta}_{\text{ext}} &= 0\\ F^{\mu\nu}_{\text{ext}}F^{\text{ext}}_{\mu\nu} &> 0\end{aligned} \tag{4.2}$$

which characterise the purely electric nature of $F^{\mu\nu}_{\text{ext}}$) and where J^{μ}_{resp} is the response to $F^{\mu\nu}_{\text{ext}}$. To this end we start once more with the BGK equation (4.4) of I where $F^{\mu\nu}$ is now replaced by $F^{\mu\nu}+F^{\mu\nu}_{\text{ext}}$. At order one in τ, $f_{(1)}(x,p)$ still has the form (4.14) of I with $F^{\mu\nu}$ also being replaced by $F^{\mu\nu}+F^{\mu\nu}_{\text{ext}}$. Multiplying this new expression for $f_{(1)}(x,p)$ by $\tau(e/m)p^{\mu}$ and integrating over the p variables we obtain

$$\tau J^{\mu}_{(1)} = -\frac{e\tau}{m}(\partial_{\lambda}E^{\mu\lambda}+\partial_{\beta}u_{\lambda}S^{\mu\beta\lambda}) - \frac{\tau e^2}{m}F^{\alpha\beta}_{\text{ext}}G^{\mu}_{\alpha\beta} \tag{4.3}$$

where the tensors $E^{\mu\lambda}$, $S^{\mu\beta\lambda}$ and $G^{\mu}_{\alpha\beta}$ are given by equations (2.7), (2.8) and (2.9) respectively. In equation (4.3) the first term is the heat flux (multiplied by e) while strictly speaking the second *is* the linear response to the external electric field $F^{\alpha\beta}_{\text{ext}}$. It follows that

$$\Lambda^{\mu\alpha\beta} = -\frac{\tau e^2}{m}G^{\mu\alpha\beta}. \tag{4.4}$$

The tensor $G^{\mu\alpha\beta}$ can be rewritten as

$$G^{\mu\alpha\beta} = \int \mathrm{d}_4p\,\frac{f_{\text{eq}}(x,p)}{p_{\lambda}u^{\lambda}}\left((p^{\beta}g^{\mu\alpha}+p^{\mu}g^{\alpha\beta}) - \frac{p^{\mu}p^{\beta}u^{\alpha}}{p_{\lambda}u^{\lambda}}\right). \tag{4.5}$$

In equation (4.5) the term involving $p^{\mu}g^{\alpha\beta}$ does not contribute to the conductivity tensor which must necessarily be antisymmetrical in those indices (α,β). Writing

$$\int \mathrm{d}_4p\, f_{\text{eq}}(x,p)\frac{p^{\beta}}{p_{\lambda}u^{\lambda}} = au^{\beta} \tag{4.6}$$

(there is no term proportional to n^{β} because of the symmetry of f_{eq} under reflexions along the direction of the magnetic field n^{β}) and

$$\int \mathrm{d}_4p\, f_{\text{eq}}(x,p)\frac{p^{\mu}p^{\beta}}{(u_{\lambda}p^{\lambda})^2} = b_1u^{\mu}u^{\beta} - b_2\pi^{\mu\beta} + b_3n^{\mu}n^{\beta} \tag{4.7}$$

one can easily see that

$$b_1 = a_1 = \int \mathrm{d}_4p\, f_{\text{eq}}(x,p). \tag{4.8}$$

Moreover, after due antisymmetrisation, one obtains

$$\Lambda^{\mu\alpha\beta} = -\frac{\tau e^2}{2m}[(b_2-b_1)(u^{\alpha}\pi^{\mu\beta}-u^{\beta}\pi^{\mu\alpha}) - (b_3-b_1)(n^{\beta}u^{\alpha}-n^{\alpha}u^{\beta})n^{\mu}]. \tag{4.9}$$

The first term in this expression for the conductivity tensor connects the transverse part of the current to the transverse electric field and thus represents the transverse conductivity

$$\sigma_{\perp} = \frac{\tau e^2}{2m}(b_2-b_1). \tag{4.10}$$

† By 'weak' we mean $|F^{\text{ext}}_{\mu\nu}|^2 \ll |F_{\mu\nu}|^2$.

The remaining term connects the longitudinal part of the current to the longitudinal electric field and thus represents the *longitudinal conductivity*

$$\sigma_{\|} = -\frac{\tau e^2}{2m}(b_3 - b_1). \tag{4.11}$$

Note that

$$b_2 = -\frac{1}{2}\int \mathrm{d}_4p\, f_{\mathrm{eq}}(x, p)\frac{\pi_{\mu\nu}p^{\mu}p^{\nu}}{(p_\lambda u^\lambda)^2}, \tag{4.12}$$

$$b_3 = \int \mathrm{d}_4p\, f_{\mathrm{eq}}(x, p)\frac{(p^\lambda n_\lambda)^2}{(p_\lambda u^\lambda)^2}. \tag{4.13}$$

Before discussing the validity of the expressions for $\sigma_{\|}$ and $\sigma_{\perp}$ we must add that unlike the classical (i.e. non-relativistic and non-quantum) case the conductivity tensor given in equation (4.9) contains only two terms related to $\sigma_{\|}$ and $\sigma_{\perp}$ and *not* a third one related to the Hall effect. This is due to the fact that equations (4.6) and (4.7) do not involve terms linear in n^α since $f_{\mathrm{eq}}(x, p)$ is invariant against the change $h \to -h$. In the classical case though the Maxwell–Boltzmann distribution obviously possesses the same invariance property, the Hall effect and the conductivity are due to an entirely different process†.

Let us now look at the validity of the above calculations.

We must first emphasise that the above conductivity tensor is only due to *collisions* and that *collective* effects are at least partly neglected. This is valid only for a dense *collision-dominated* plasma. If collective effects are also to be taken into account, since we are concerned only with a *linear* response, it is sufficient to add to $\Lambda^{\mu\alpha\beta}$ a term which has exactly the same structure as the non-quantum relativistic expression (Hakim and Mangeney 1968, 1971, Hakim and Heyvaerts 1977). In fact at moderately high densities the quantum plasma is often completely collision dominated otherwise at ultra-high densities its behaviour again becomes collective (see I, § 5).

In the non-relativistic case it is generally argued (Kahn and Frederikse 1969, Argyres and Adams 1956, Argyres 1960) that since the nature of the motion of a charged particle is different in the case of crossed or parallel electric and magnetic fields, it seems necessary to have different treatments for the calculations of the transverse and longitudinal conductivities. In our relaxation time model this might signify that it would perhaps be necessary to introduce two relaxation times, $\tau_{\|}$ and $\tau_{\perp}$, to be evaluated by two different methods. In fact a closer analysis of the usual calculation of the conductivities of a quantum plasma embedded in a magnetic field (see the references given at the beginning of this paragraph) shows that: (i) $\sigma_{\|}$ can be calculated with the *diagonal elements* of the density matrix only, thereby allowing the use of pseudo-distribution functions depending on $p_{\|}$, one for each Landau level; and (ii) $\sigma_{\perp}$ requires a different treatment essentially because only *off-diagonal elements* (diagonal in the quantum number $p_{\|}$ and off-diagonal as to n) of the density matrix are involved in the calculations and hence no pseudo-distribution function can be used to relax towards equilibrium. In this paper, however, we have used a quantum distribution which is essentially *equivalent* to the complete density matrix so that one might think that our treatment could be valid.

† In this paper the dependence of our transport coefficients occurs only via the equilibrium distribution. This is a typically quantum effect. In the classical (non-quantum) case, it occurs via the Lorentz force term, which is linear in the magnetic field.

Unfortunately the situation is by no means so simple since $f_{(1)}$, which approximates f, is a functional of f_{eq} itself. On the other hand f_{eq} has been calculated neglecting interactions, and therefore is linked to the diagonal part of the density matrix and *not* to its off-diagonal elements. It then follows that our treatment of $\sigma_\perp$ is probably not correct although the evaluation of the relaxation time τ certainly involves the whole density matrix. As to $\sigma_\parallel$ our treatment is certainly as good as all other treatments based on a relaxation time approximation (Canuto and Chiu 1969, 1970).

This brings us to the third problem, a conceptual one: Is the relaxation towards equilibrium actually exponential? Of course this problem exists in all relaxation time models and depends on the particular physical situation under study. Anyway it is generally extremely difficult to solve.

Let us finally add one more remark. If the applied electric field is not weak† then it is necessary to use an equilibrium quantum distribution taking account of both $F^{\mu\nu}$ and $F^{\mu\nu}_{ext}$ (Dominguez Tenreiro and Hakim 1977b). However, in such a case the response of the system ceases to be linear.

5. Discussion and conclusion

Let us now summarise the results obtained in this paper (the approximations used have been discussed in I).

(1) We have obtained *forms* for relativistic hydrodynamical equations in the presence of a magnetic field and, as a consequence, the general expressions for the various transport coefficients. In the definitions of the latter we have encountered ambiguities inherent to the presence of magnetic fields, ambiguities which also exist in the non-quantum and non-relativistic case. Here we must point out that the expressions obtained for the transport coefficients are still valid if we replace $f_{eq}(p)$ by any pseudo-equilibrium function towards which the gas relaxes in a proper time $\tilde{\tau}$ (much smaller than τ, the relaxation time towards $f_{eq}(p)$: $\tilde{\tau}$ then should replace τ everywhere) provided the only four-vectors on which it *depends* are u^μ and h^μ, β being replaced by a parameter characterising for instance the energy content of the gas. Finally we have given expressions for the dc conductivities (longitudinal and transverse) whose validities depend strongly on the validities: (i) of the relaxation time model itself; and (ii) of the evaluation of the relaxation time.

(2) The limit of a vanishing magnetic field can be obtained easily once we have noticed that $f_{eq}(p)$ tends to an *isotropic* function

$$\lim_{h\to 0} f_{eq\,(aniso)}(p) = f_{eq(iso)}(p),$$

the isotropic function being the usual relativistic equilibrium quantum distribution (up to inessential δ terms, which accounts for its four dimensional normalization). For instance, one then has

$$\lim_{h\to 0} P_\parallel = \lim_{h\to 0} P_\perp = P$$

(P being the pressure of the ordinary relativistic Fermi gas). Let us show how these remarks apply to the case of the thermal conductivity coefficients. A glance at

† See footnote to page 1537.

equations (2.38) and (2.39) shows that in order to obtain the same limit for $\lambda_{\parallel}$ and $\lambda_{\perp}$ when h tends to zero, it is necessary to have

$$\lim_{h\to 0} E_2 = \lim_{h\to 0} E_3.$$

In fact we have

$$\lim_{h\to 0} E_3 = -\int \frac{d_4p}{p_0} p_3^2 \lim_{h\to 0} f_{eq}(p)$$

$$\lim_{h\to 0} E_2 = -\frac{1}{2}\int \frac{d^4p}{p_0}(p_1^2+p_2^2) \lim_{h\to 0} f_{eq}(p)$$

$$= -\frac{1}{2}\int \frac{d_4p}{p_0} 2p_1^2 f_{eq\,(iso)}(p) = \lim_{h\to 0} E_3$$

in a Lorentz frame where $u^{\mu} = (1, 0, 0, 0)$ and $n^{\mu} = (0, 0, 0, 1)$. The other limits can be obtained in a similar manner although in a slightly more complicated way.

(3) Let us now explain how the conditions treated in our work differ from those considered by Anderson (1977) in his calculations of the transport coefficients of the relativistic degenerate plasma.

Firstly we treat the case of a strong magnetic field (i.e. $\Omega_B\tau \gg 1$, $n^* \sim 1$) whereas Anderson deals with the moderately strong field case ($\Omega_B\tau \gg 1$, $n^* \gg 1$) (Canuto and Ventura 1977 and references quoted therein).

Consequently we used a magnetic-field-dependent quantum distribution while the one used by Anderson is the usual relativistic Fermi–Dirac distribution. A second reason, which is partially a consequence of the latter, is the following. The magnetic field dependence (for instance, in direction) of the quantities relevant to the calculation of the transport coefficient (i.e. $J^{\mu}_{(1)}$, $T^{\mu\nu}_{(1)}$) occurs—in Anderson's approach—only through the term

$$F^{\mu\nu}p_{\nu}\frac{\partial}{\partial p^{\mu}}f,$$

while in our treatment it occurs *only* through the equilibrium distribution function. Consequently our transport coefficients are merely of *quantum origin* while Anderson's have their classical counterparts (when $h \neq 0$). At this point it remains to explain the reason why we do not obtain Anderson's terms in addition to our own. This is due essentially to the constraint (3.10) of I which prevents the occurrence of terms linear in $F^{\mu\nu}$ in the off-equilibrium parts of the four-current and of the energy–momentum tensor. The constraint (3.10) of I—necessary in our case—does not exist in Anderson's article because of the total independence of u^{μ} (occurring in its equilibrium distribution) and $F^{\mu\nu}$ which contains a small electric part. Also note that the approaches of these two papers are quite different in that we have used the Chapman–Enskog expansion rather than the variational treatment of Anderson and the BGK collision term rather than the Boltzmann collision term.

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